Comment on 'On the uncertainty relations and squeezed states for the quantum mechanics on a circle'

2003 J. Phys. A: Math. Gen. 36 2197

(http://iopscience.iop.org/0305-4470/36/8/316)

Institute of Physics Publishing | Journal of Physics A: Mathematical and General

J. Phys. A: Math. Gen. **36** (2003) 2197–2202 | PII: S0305-4470(03)53176-6

COMMENT

Comment on 'On the uncertainty relations and squeezed states for the quantum mechanics on a circle'

D A Trifonov

Institute for Nuclear Research, 72 Tzarigradsko chaussee, 1784 Sofia, Bulgaria

Received 4 September 2002
Published 12 February 2003
Online at stacks.iop.org/JPhysA/36/2197

Abstract

It is shown by examples that the position uncertainty on a circle, proposed recently by Kowalski and Rembieliński (2002 *J. Phys. A: Math. Gen.* **35** 1405) is not consistent with the state localization. We argue that the relevant uncertainties and uncertainty relations (URs) on a circle are those based on the Gram–Robertson matrix. Several of these generalized URs are displayed and related criteria for squeezed states are discussed.

PACS numbers: 03.65.−w, 45.50.Dv

1. In the recent paper [1] the problem of a relevant uncertainty relation (UR) for the angular momentum and the angle variables of a particle on a circle was discussed and a new UR was proposed. Noting a contradiction in the previously obtained UR [2] the authors define new quantities $\tilde{\Delta}^2(\hat{\varphi})$ and $\tilde{\Delta}^2(\hat{J})$ ($\hat{\varphi} = \varphi$, $\hat{J} = -\mathrm{i}\,\mathrm{d}/\mathrm{d}\varphi$) as measures for the uncertainty of the angle φ and the angular momentum J and suggest the inequality

$$\tilde{\Delta}^2(\hat{\varphi}) + \tilde{\Delta}^2(\hat{J}) \geqslant 1. \tag{1}$$

The quantities $\tilde{\Delta}^2(\hat{\varphi})$ and $\tilde{\Delta}^2(\hat{J})$ are defined as [1] (note a change in notation: $\Delta^2 \to \tilde{\Delta}^2$)

$$\tilde{\Delta}^2(\hat{\varphi}) = -\tfrac{1}{4}\ln|\langle U^2\rangle|^2 \qquad \tilde{\Delta}^2(\hat{J}) = \tfrac{1}{4}\ln(\langle \mathrm{e}^{-2\hat{J}}\rangle\langle \mathrm{e}^{2\hat{J}}\rangle) \tag{2}$$

where $U = \exp(\mathrm{i}\hat{\varphi})$. The authors of [1] find that for the eigenstates $|z\rangle$ of the operator $Z = \exp(-\hat{J} + 1/2)U$, $Z|z\rangle = z|z\rangle$ (the 'genuine coherent states (CSs) for a quantum particle on a circle' [3]) both quantities (2) equal 1/2, and suggest that $\tilde{\Delta}^2(\hat{\varphi})$ and $\tilde{\Delta}^2(\hat{J})$ obey inequality (1) in any state. Henceforth, the quantities $\tilde{\Delta}^2(\hat{\varphi})$ and $\tilde{\Delta}^2(\hat{J})$ should be referred to as Kowalski–Rembieliński uncertainties (KR uncertainties), the UR (1) as KR UR and $|z\rangle$ as Kowalski–Rembieliński–Papaloucas CSs (KRP CSs).

2. Next we shall demonstrate that the KR uncertainty $\tilde{\Delta}^2(\hat{\varphi})$ is not consistent with the state localization on a circle. For this purpose we compare the φ-probability distributions $p_\psi(\varphi)$ (defined as $p_\psi(\varphi) = |\psi(\varphi)|^2 = |\langle\varphi|\psi\rangle|^2$) in KRP CSs with φ-distributions in certain states with squeezed $\tilde{\Delta}^2(\hat{\varphi})$. The quantity $\tilde{\Delta}^2(\hat{\varphi})$ is called [1] squeezed if it is less than 1/2. The authors of [1] constructed a family of such squeezed states $|z\rangle_s$ as eigenstates of the operator $Z(s) = \exp(-s\hat{J} + s/2)U = \exp(\mathrm{i}\varphi - s\hat{J})$, where s is a positive parameter. Here we shall

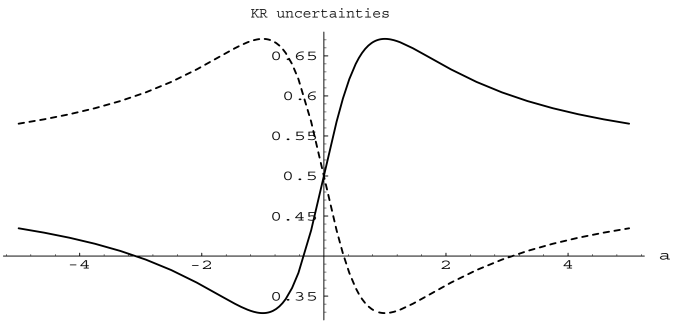

Figure 1. KR uncertainties $\tilde{\Delta}^2(\hat{\varphi})$ (solid line) and $\tilde{\Delta}^2(\hat{J})$ (dashed line) in cat states $|z=1,a\rangle$ as functions of a. $\tilde{\Delta}^2(\hat{\varphi})$-squeezing is maximal around $a=-1$.

consider $\tilde{\Delta}^2(\hat{\varphi})$-squeezed states of the form of eigenstates $|z,a\rangle$ of the squared operator Z^2. These are defined as macroscopic superpositions of $|z\rangle$ and $|-z\rangle$ (Schrödinger cat states on a circle),

$$|z,a\rangle = N(z,a)(|z\rangle + a|-z\rangle) \tag{3}$$

where a is a complex parameter, and the normalization constant $N(z,a)$ takes the form

$$N(z,a) = [1+|a|^2+2\langle z|-z\rangle \operatorname{Re} a]^{-1/2}. \tag{4}$$

The scalar product of two CSs is [1] $\langle z|\eta\rangle = \theta_3((\mathrm{i}/2\pi)\ln(z^*\eta), \mathrm{i}/\pi)$, where $\theta_3(x,y)$ is the Jacobi theta-function. The states $|z, a=\pm 1\rangle \equiv |z;\pm\rangle$ should be called even/odd CS on a circle.

On the states $|z,a\rangle$ the quantities $\langle U^2\rangle$, $\langle \exp(2J)\rangle$, $\langle \exp(-2J)\rangle$ in (2) take the form

$$\langle a,z|U^2|z,a\rangle = N^2(z,a)(\langle z|U^2|z\rangle + |a|^2\langle -z|U^2|-z\rangle + a\langle z|U^2|-z\rangle + a^*\langle -z|U^2|z\rangle) \tag{5}$$

$$\langle a,z|\,\mathrm{e}^{\pm 2J}|z,a\rangle = N^2(z,a)(\langle z|\,\mathrm{e}^{\pm 2J}|z\rangle + |a|^2\langle -z|\,\mathrm{e}^{\pm 2J}|-z\rangle + 2\operatorname{Re}(a\langle z|\,\mathrm{e}^{\pm 2J}|-z\rangle)) \tag{6}$$

where $\langle z|U^2|z\rangle$, $\langle z|\,\mathrm{e}^{2J}|z\rangle$ and $\langle z|\,\mathrm{e}^{-2J}|z\rangle$ are given by z/ez^*, $e/|z|^2$ and $e|z|^2$ respectively [1]. Substituting (5) and (6) in (2) we obtain explicit formulae for KR uncertainties in $|z,a\rangle$.

From formulae (2), (5) and (6) we find that $\tilde{\Delta}^2(\hat{\varphi})$-squeezing occurs in many superpositions $|z,a\rangle$, in particular in $|z;\pm\rangle$ (see figure 1). In the odd state $|1;-\rangle$, corresponding to the solid line minimum in figure 1, we find $\tilde{\Delta}^2(\hat{\varphi}) \approx 0.33$, which is considerably less than the value $1/2$ of $\tilde{\Delta}^2(\hat{\varphi})$ in CSs $|z\rangle$. One should expect that the φ-distribution, corresponding to wavefunctions with squeezed 'position uncertainty' $\tilde{\Delta}^2(\hat{\varphi})$, is better localized on the circle than the non-squeezed CS. Unfortunately it is not the case with $\tilde{\Delta}^2(\hat{\varphi})$-squeezed states from the family $\{|z,a\rangle\}$. This inconsistency is demonstrated in figure 2 on the example of cat state $|1;-\rangle$. As one can see from figure 2 the $\tilde{\Delta}^2(\hat{\varphi})$-squeezed state $|1;-\rangle$ is much worse localized than the non-squeezed CS $|z\rangle$ ($p(\varphi)$-distributions of $|z\rangle$ with different z approximately coincide up to a translation). Therefore, the quantity $\tilde{\Delta}^2(\hat{\varphi})$ is not a proper measure of the position uncertainty, and inequality (1) could hardly be qualified as a relevant uncertainty relation on a circle.

Let us note that $|z,a\rangle$ saturate inequality (1) with unequal $\tilde{\Delta}^2(\hat{\varphi})$ and $\tilde{\Delta}^2(\hat{J})$, the case of $z=1$ and real a being demonstrated in figure 1. However, the whole range of validity of (1) is not yet clarified. Nevertheless it might be interesting to note that in the variety of states on the real line a similar inequality holds, i.e. $\tilde{\Delta}^2(\hat{x}) + \tilde{\Delta}^2(\hat{p}) \geqslant 1$, where $\hat{x}$ and $\hat{p}$ are position and momentum operators, respectively.

3. The above remarks naturally raise again the questions about the position and angular momentum uncertainties and the relevant *uncertainty relations (URs) on a circle*. In my

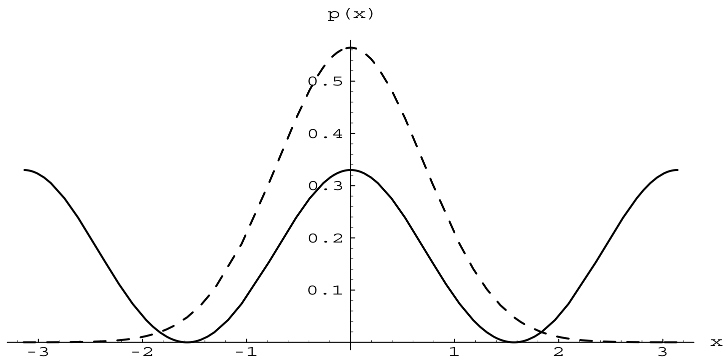

Figure 2. The distributions $p(x)$ as functions of the angle $\varphi \equiv x$ for $\tilde{\Delta}^2(\hat{\varphi})$-squeezed state $|1; -\rangle$ (solid line) and for non-squeezed CS $|z = 1\rangle$ (dashed line). CS $|1\rangle$ is better localized than $|1; -\rangle$.

opinion the most suitable URs for n observables X_i and one state $|\psi\rangle$ on a circle are those based on the Gram–Robertson matrix $G = \{G_{ij}\}$ of the form [4] $(i, j = 1, \ldots, n; n = 1, 2, \ldots)$

$$G_{ij}(\psi) = \langle (X_i - \langle X_i \rangle)\psi | (X_j - \langle X_j \rangle)\psi \rangle. \tag{7}$$

The more informative notation $G(\vec{X}; \psi)$ and $G_{X_iX_j}(\psi)$ $(\vec{X} = X_1, X_2)$ for this matrix and its elements should also be used. The generalized covariances ${}_g\Delta X_i X_j(\psi)$ of X_i and X_j in $|\psi\rangle$ are defined [4] as the symmetric part S_{ij} of G_{ij} (for the case of $n = 2$ see also [5, 6])

$${}_g\Delta X_i X_j(\psi) := S_{X_iX_j}(\psi) = \mathrm{Re}\langle (X_i - \langle X_i \rangle)\psi | (X_i - \langle X_i \rangle)\psi \rangle. \tag{8}$$

The diagonal elements S_{ii} are defined as generalized variances $({}_g\Delta X_i)^2$ of X_i.

Since G is non-negative all the characteristic coefficients of its symmetric part $S = (G + G^T)/2$ are not less than the corresponding characteristic coefficients of its antisymmetric part $A = (G - G^T)/2\mathrm{i}$. These inequalities are called generalized *characteristic URs* [4]. The senior characteristic UR reads

$$\det S(\vec{X}; \psi) \geqslant \det A(\vec{X}; \psi). \tag{9}$$

In the simplest case of $n = 2$ this UR is displayed as $S_{11}S_{22} - S_{12}^2 \geqslant A_{12}^2$. It can also be written in the shorter form $\det G \geqslant 0$, and displayed in terms of the generalized covariances as

$$({}_g\Delta X_1)^2({}_g\Delta X_2)^2 \geqslant ({}_g\Delta X_1 X_2)^2 + (\mathrm{Im}\langle (X_1 - \langle X_1 \rangle)\psi | (X_2 - \langle X_2 \rangle)\psi \rangle)^2. \tag{10}$$

The sum of the two terms on the right-hand side of (10) is just the squared absolute value of G_{12}, i.e. we have ${}_g\Delta X_{1g}\Delta X_2 \geqslant |G_{12}(\psi)|$.

When the actions of X_iX_j on $|\psi\rangle$ are correctly defined (normal cases) the above Gram matrix coincides [4] with the Robertson one [7]: its antisymmetric part A_{ij} reduces, up to a factor, to the mean commutator, $A_{ij} = -(\mathrm{i}/2)\langle [X_i, X_j] \rangle$, and its symmetric part takes the familiar form of the standard uncertainty matrix $\sigma(\vec{X}; \psi)$. (The element $\sigma_{ij} = \langle X_iX_j + X_jX_i \rangle/2 - \langle X_i \rangle \langle X_j \rangle \equiv \Delta X_i X_j$ is the standard covariance of X_i and X_j, and $\sigma_{ii} = \Delta X_i X_i \equiv (\Delta X_i)^2$ is the variance of X_i. $(\Delta X)^2$ should not be confused with the KR quantity $\tilde{\Delta}^2(X)$.) Under these conditions inequality (9) takes the form of Robertson UR for n observables [4, 7, 8], and (10) coincides with the Schrödinger (or Schrödinger–Robertson) UR [9] (for a review on this UR and its minimization states see, e.g., [8]).

The generalized form of the less precise Heisenberg UR reads $({}_g\Delta X_1)^2({}_g\Delta X_2)^2 \geqslant (\mathrm{Im}\, G_{12})^2$, and it again follows from the more precise one (10). For a similar generalization

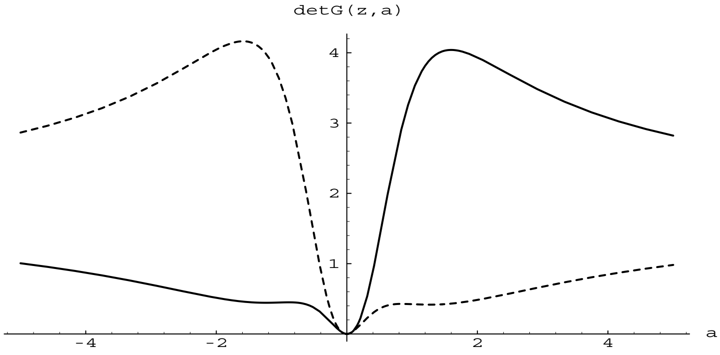

Figure 3. Illustration of the generalized Schrödinger uncertainty relation (10) in the superpositions $|z, a\rangle$: $\det G = (\Delta\varphi)^2(\Delta J)^2 - |G_{J\varphi}|^2$ as a function of a for $z = 0.4$ (solid line) and $z = 1$ (dashed line).

see also [5, 6]. (Please note that in some papers, e.g. [10], no distinction is made between Schrödinger and Heisenberg URs, both being named after Heisenberg.)

Thus, in the special cases when $X_i X_j|\psi\rangle$ are not properly defined one should resort to generalized Schrödinger UR (10) (for two observables), and to (9) (for several observables). The position and the momentum observables of a particle on a circle represent such a special case, since $\hat\varphi\langle\varphi|\psi\rangle = \varphi\langle\varphi|\psi\rangle$ is not 2π-periodic and $\hat J$ is not Hermitian on such functions. Another special case of interest is particle motion on the sphere.

Figure 3 illustrates the generalized UR (10) in the case of $X_1 = \hat J$ and $X_2 = \hat\varphi$ and states $|z, a\rangle$ (particle on a circle), where $\det G(z, a) = \det G(\vec X; z, a)$ are plotted as functions of real a for $z = 0.4$ (solid line) and $z = 1$ (dashed line). In these states the generalized covariance ${}_g\Delta\varphi J = \mathrm{Re}\, G_{J\varphi}$ vanishes, also ${}_g\Delta\varphi = \Delta\varphi$, ${}_g\Delta J = \Delta J$, so that here we have $\det G = (\Delta\varphi)^2(\Delta J)^2 - (\mathrm{Im}\, G_{J\varphi})^2 \geqslant 0$. The minimal value of $\det G$ in figure 3 is different from zero (it is about 0.000 17).

Unlike $\tilde\Delta^2(\hat J)$ and $\tilde\Delta^2(\hat\varphi)$, the variances $(\Delta J)^2$ and $(\Delta\varphi)^2$ are in good correspondence with the angular momentum and position localization on a circle. For example, φ-distributions for CSs $|z\rangle$ with $z = 0.4$, 1 are practically the same (see figure 2), and the variances $(\Delta\varphi)^2$ in these CS are almost equal: in $|z = 0.4\rangle$ $(\Delta\varphi)^2 = 0.500\,55$, and in $|z = 1\rangle$ $(\Delta\varphi)^2 = 0.500\,64$. In the worse localized cat state $|1; -\rangle$ (see figure 2) the variance $(\Delta\varphi)^2$ takes the larger value of 3.813.

We have to warn that one has to be careful about the correspondence between $\Delta\varphi$-squeezing and localization of the wavefunction $\langle\varphi|\psi\rangle$: in view of the identification of points φ and $\varphi + 2\pi$ the mean values $\langle\varphi\rangle$, $\langle\varphi^2\rangle$ should be calculated by integration from $\varphi_0 - \pi$ to $\varphi_0 + \pi$, where φ_0 is the centre of the wave packet (i.e. φ_0 is the most probable value of φ). In this way we find that both standard deviations $\Delta\varphi$ and ΔJ in KRP CSs $|z\rangle$ show very small oscillations around the value of 1/2. So, the family $\{|z\rangle\}$ consists of almost minimum uncertainty states on the circle.

4. The *minimization states* (intelligent, or minimum-uncertainty states) of the generalized UR (10) for X_1 and X_2 should be eigenstates of a real or complex combination $\mu X_1 + \nu X_2$. In the case of the particle on a circle and $X_1 = \hat J$ and $X_2 = \hat\varphi$ the 2π-periodicity condition on the wavefunctions $\psi(\varphi + 2\pi) = \psi(\varphi)$ should be imposed (some authors admit exceptions [6]). This restriction rules out all solutions of the eigenvalue equation $(\mu\hat J + \nu\hat\varphi)|\psi\rangle = z|\psi\rangle$, except for the eigenstates $\psi_m(\varphi)$ of $\hat J$, $\psi_m(\varphi) = (1/\sqrt{2\pi})\exp(\mathrm{i}m\varphi)$. For $\psi_m(\varphi)$ we have

$\Delta J = 0$, $\Delta\varphi = \sqrt{\pi}$, $G_{J\varphi} = 0$, so that the equality in UR (10) reads $0 = 0$. None of the states $|z, a\rangle$ and $|z\rangle_s$ minimize inequalities (10), although the deviations in the case of CSs $|z\rangle$ are very small, as is illustrated in figure 3 at $a = 0$.

In order to define *squeezed states* on the circle let us recall that for the particle on the real line these states are defined by means of one of the two inequalities $(\Delta x)^2 < |\langle[x, p]\rangle|/2 = 1/2$, or $(\Delta p)^2 < |\langle[x, p]\rangle|/2 = 1/2$. Since $\mathrm{Im}\, G_{12}(\psi)$ is a generalization of the mean commutator $(-\mathrm{i}/2)\langle[X_1, X_2]\rangle$ one can define X_1–X_2 squeezed states more generally as states for which

$$({}_g\Delta X_i)^2\rangle|\mathrm{Im}\, G_{12}(\psi)| \qquad i = 1 \text{ or } 2. \tag{11}$$

This is a generalization of the well-known Eberly–Wodkiewicz criterion for squeezed states. It is, however, a relative criterion, since the 'generalized mean commutator' $|\mathrm{Im}\, G_{12}(\psi)|$ may take, in general, values from 0 to ∞. Another stronger criterion for squeezed states is suggested by the observation that on the real line (and for the one mode electromagnetic field) $1/2$ is the minimal value that two variances $(\Delta x)^2$ and $(\Delta p)^2$ can take simultaneously. Therefore, we can define X_1–X_2 squeezed states more generally as states for which one of the following two inequalities holds,

$$({}_g\Delta X_i)^2\rangle\Delta_0^2 \qquad i = 1 \text{ or } 2 \tag{12}$$

where Δ_0^2 is the minimal value that the two generalized variances can take simultaneously. For incompatible observables $\Delta_0 > 0$. It is plausible that $2\Delta_0^2$ is the lower limit of the sum of two variances,

$$(\Delta X_1)^2 + (\Delta X_2)^2 \geqslant 2\Delta_0^2. \tag{13}$$

If the eigenstates of $X_1 + \mathrm{i}X_2$ (or $X_1 - \mathrm{i}X_2$) exist (canonical observables, spin and quasi-spin components etc), then Δ_0^2 is equal to the minimal value of $|\mathrm{Im}\, G_{J\varphi}(\psi)|$ within *these* eigenstates, and (13) is rigorously valid [8]. If eigenstates of $X_1 \pm \mathrm{i}X_2$ do not exist, the critical quantity Δ_0 should be estimated by different methods. The case of $X_1 = \hat{J}$ and $X_2 = \hat{\varphi}$ is such a special case, since 2π-periodic eigenfunctions of $\hat{\varphi} \pm \mathrm{i}\hat{J}$ do not exist. Numerical considerations suggest that in this case $\Delta_0^2 \approx 0.5$ (more precisely $\approx 0.499\,99$), which is the minimal value that $(\Delta\varphi)^2$ and $(\Delta J)^2$ take simultaneously in CSs $|z\rangle$.

It turned out that both criteria (11) and (12) can be satisfied in many states from the families $\{|z, a\rangle\}$ and $\{|z\rangle_s\}$. Squeezing of $\Delta\varphi$ in $|z, a\rangle$ is not very strong, while in $|z\rangle_s$ it can be arbitrarily strong.

Of course $|z\rangle$ are exact Heisenberg intelligent states for the Hermitian components X, Y of Z. However, neither ΔX nor ΔY is in a satisfactory correspondence with the localization on a circle, as one can easily check for the example of cat states $|z; \pm\rangle$.

In conclusion we note that the above-described scheme can be extended to represent correct generalized URs for several observables and (several) mixed states as well [4].

Note added in proof. For the sake of completeness, I have to note that coherent states on a circle have been introduced (in more general notations) by S De Bievre and J Gonzalez in 1993 [2].

References

[1] Kowalski K and Rembieliński J 2002 *J. Phys. A: Math. Gen.* **35** 1405–14 (*Preprint* quant-ph/0202070)
[2] Gonzáles J A and del Olmo M A 1998 *J. Phys. A: Math. Gen.* **31** 8841
[3] Kowalski K, Rembieliński J and Papaloucas L C 1996 *J. Phys. A: Math. Gen.* **29** 4149

[4] Trifonov D A 2000 *J. Phys. A: Math. Gen.* **33** L299 (*Preprint* quant-ph/0005086)
Trifonov D A 2001 *J. Phys. A: Math. Gen.* **34** L75
[5] Chisolm E D 2001 *Am. J. Phys.* **69** 368 (*Preprint* quant-ph/0011115)
[6] Dumitru S 2002 On a subject of diverse improvisations: the uncertainty relations on a circle *Preprint* quant-ph/0206009
[7] Robertson H P 1934 *Phys. Rev.* **46** 794
[8] Trifonov D A 2001 *World Phys.* **24** 107 (*Preprint* physics/0105035)
Trifonov D A 2000 *J. Opt. Soc. Am.* A **17** 2486
[9] Schrödinger E 1930 *Sitz. Preus. Acad. Wiss. (Phys.-Math. Klasse)* **19** 296–303
Robertson H P 1930 *Phys. Rev.* **35** 667 (abstract only)
[10] Agarwal G S 2002 *Fortschritte Phys.* **50** 575 (*Preprint* quant-ph/0201098)